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Global rate equation description of a laser

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Abstract. The global integro-differential rate equations describing a multimode laser are analyzed. Expressions for the relaxation oscillation frequencies and their damping rates in the single-mode and two-mode regimes are obtained without specifying either the cavity geometry or the longitudinal pump profile. On the same level of generality, we prove the existence of universal relations relating the peaks of the power spectra in the two-mode regime. For a Fabry-Perot with arbitrary longitudinal pump profile, series expansions of all the physical functions are derived in powers of the pump moments. These moments are averages of the pump profile over cavity modes at linear combinations of the lasing frequencies and their harmonics. These results apply to end-pumped and/or partially filled lasers. For a single mode Fabry-Perot laser, we prove that the contribution to the steady state intensity from the lasing mode varies from 75% close to the lasing threshold to zero at high intensity. The remainder comes from the harmonics of the lasing mode. Analyzing the steady state single mode intensity equation in terms of the pump gratings, we prove that close to the lasing threshold only the space average of the pump and its grating oscillating at twice the lasing wave number do not vanish. This provides a hint towards the justification of the usual modal rate equations which retain only these two functions in the dynamical evolution of a laser. For a Fabry-Perot with constant pump profile, an exact expression for the upper boundary of the stable single mode regime is derived. In that two-mode regime, we prove that there is a critical value of the pump at which the ratio of the two relaxation oscillation frequencies is 2, leading to an internal resonance.

PACS. 42.55.Ah General laser theory – 42.65.Sf Dynamics of nonlinear optical systems; optical instabilities, optical chaos and complexity, and optical spatio-temporal dynamics

1 Introduction

The purpose of this paper is to present a study of the global rate equations describing a homogeneously broadened multimode laser. In reduced variables, these equations are [1]

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{I}(p,\tau)=\kappa_p\mathcal{I}(p,\tau)\left\{-1+\frac{\gamma_p}{L}\int_0^L f_p(z)\mathcal{J}(z,\tau)\mathrm{d}z\right\}, \tag{1}$$

$$\frac{\partial}{\partial\tau}\mathcal{J}(z,\tau)=w(z)-\mathcal{J}(z,\tau)\left\{1+\sum_p f_p(z)\mathcal{I}(p,\tau)\right\}, \tag{2}$$

where $f_p(z)\equiv|\phi_p(z)|^2$ and $\phi_p(z)$ is a lasing cavity eigenmode, $\mathcal{I}_p(\tau)$ is the intensity of mode p normalized to its saturation intensity, time and time constants have been scaled to the population inversion decay time, κ_p and γ_p are, respectively, the decay rate and the linear gain of mode p, $\mathcal{J}(z,\tau)$ is the population inversion and $w(z)$ is the longitudinal pump profile. There are N modes labelled by the index $p=1,2,\ldots,N$. The cavity length is L. In this formulation, transverse effects are neglected.

Up to now, most studies of equations (1, 2) have been based on modal expansions of the population inversion. This leads to an infinite hierarchy of equations which is truncated without much justification to yield a finite set of purely differential equations. The most popular truncation scheme, leading to the simplest rate equations, was proposed by Tang, Statz, and deMars [2]. The TSD rate equations couple the modal intensities to the population inversion averaged over the cavity length and to the population gratings at the optical frequencies. The population gratings are related simply to the averages of $\mathcal{J}(z,\tau)$ over the cavity modes. The TSD rate equations form the minimal set of modal equations that can be derived from the global equations with $\phi_p(z)=\sqrt{2}\sin(k_pz)$ and which include the effect of population grating. These rate equations rely on a set of assumptions, some of which were sensible in the early sixties but are no longer necessarily applicable to most lasers: the amplifying medium fills the resonant cavity, the longitudinal pump profile (*i.e.*, the pump profile along the optical axis of the cavity) is constant, and the resonator is a linear Fabry-Perot resonator. By linear, we mean a Fabry-Perot cavity whose optical axis is a straight line, as opposed to standing wave

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configurations where the cavity has two (or more) arms connected by a mirror or a grating. Such cavities have different polarization properties which affect the performance of the laser if a birefringent element is introduced.

Among the recent developments in laser physics is the emergence of new laser cavities. They include high-Q small-volume optical resonators such as Nd:YAG spheres [3], Nd:glass micro-spheres [4], micro-cylinder semiconductor lasers with bow-tie whispering gallery modes in asymmetric resonators [5], and recently 8 μm hexagonal cavities operating on whispering gallery modes in zeolite-dye microlaser [6]. These laser cavities are often not reducible to a linear Fabry-Perot [7,8], which indicates the need for a laser theory that does not rely explicitly on the cavity mode properties. In this context, a problem which will not be discussed here is whether a deterministic rate equation description is valid or not. In a recent paper [9], it has been argued that for a thresholdless microlaser, deterministic rate equations are no longer the adequate framework because spontaneous noise dominates the laser dynamics at low pump rates. This certainly places a physical constraint on the validity of the rate equations, of either the modal or the global varieties. Still, the results of [9] should be taken with some care since they also rely heavily on a truncation procedure similar to the TSD scheme.

Many solid-state lasers today are end-pumped by another laser, often a diode laser. Therefore, the longitudinal profile of the pump inside the resonator is not constant but obeys Beer-Lambert's law $w(z) \sim \exp(-z/\alpha)$ at low power. The absorption length α is strongly frequency-dependent and the pump normally operates at a frequency quite different from the lasing frequency. Thus α is not related in any simple way to the lasing characteristics of the pumped medium. In addition, even if the pumping is orthogonal to the laser axis, it may vary along the cavity length. Early attempts to generalize the TSD equations to include the longitudinal pump variations have been restricted by some other assumptions, such as a pump profile which can be described in terms of only a space average and one long wavelength spatial Fourier component, and also the replacement of the saturation by a cubic nonlinearity [10–12]. Recently, a systematic analysis of the moment expansion of the global rate equations led to an extension of the TSD equations [13]. It retains the long wavelength moments of the pump and of the population inversion. However, it is a recent experimental work on Nd:YAG lasers [14] that has shown most clearly the importance of the longitudinal pump profile. These authors have shown that the ordering of the modes according to their intensity does not match the ordering of the modal linear gains: the modal intensities cross each other many times as the pump increases, a property which is incompatible with the TSD equations. They also give hints that the exponential decay of the pump along the cavity axis explains qualitatively the experimental results. This problem was also discussed in a recent analysis of the $LiNdP_4O_{16}$ microchip laser dynamical properties [15]. However, in that paper exponential decay of the pump and multilevel configuration were analyzed simultaneously so that it is not possible to determine the consequences of each effect separately.

Another fundamental issue which is not accounted for by the TSD equations is the role of the filling factor. The question here is to describe a laser whose cavity is only partially filled by the amplifying medium. This problem was tackled in [16] in a TSD-like formulation. Recently, the filling factor problem has been investigated more systematically for a Nd:YAG laser in the multimode regime [17]. In that paper, experimental results are reported and compared with a numerical simulation of the global equations. The main conclusion is that the number of modes and their dynamical properties such as the relaxation oscillation frequencies critically depend on the filling factor. For instance, variations of the filling factor may lead to the suppression of modes with linear gains larger than those of the oscillating modes. Here again, the ordering of the modes according to their intensity does not match the ordering of the modal linear gains.

There are at least three more reasons for which a study of the global rate equations is necessary. One reason is that the extension of the TSD equations [13,18] predicts a self-pulsing threshold in the multimode regime if the pump is sufficiently inhomogeneous spatially. Analytic selfpulsing conditions have been derived for two- and three-mode lasers. This is in complete contradiction with the TSD equations for which the only possible instability is the appearance of a new lasing mode. Another reason is the unrealistic property that the number of relaxation oscillation frequencies and damping rates increases without limit as the number of moments which is retained in the hierarchy of evolution equations increases. Finally, in a recent paper [19], we have proved that a general antiphase theorem can be derived from equations (1, 2). In this context, it appeared clearly that assessing the role of the filling factor is not a problem if the theory is developed without any specification of the longitudinal pump profile and the results are expressed in terms of averages of the pump.

In this paper, we present results derived from the N-mode global rate equations (1, 2). However, explicit properties have been obtained only for single and two-mode lasers. In addition, an attempt is made to derive as many results as possible that are independent of the cavity mode and the longitudinal pump profile. This paper is organized as follows. In Section 2, we analyze the steady state modal intensities. This section is subdivided in three parts: in Section 2.1, we study in detail the properties of the single mode intensity. In Section 2.2 we discuss a few properties of the two-mode steady state intensities. In Section 2.3 we derive an exact equation for the boundary separating the single mode regime from the two-mode regime. In this section, most expressions hold for an arbitrary pump profile but all results are obtained for a linear Fabry-Perot. In Section 3, we analyze the evolution equations linearized around the steady state to determine the laser stability. This section is also subdivided in three parts: in Section 3.1, we obtain the relaxation oscillation frequencies and their damping rates for a single mode laser with arbitrary cavity mode and pump profile. In Section 3.2, the relaxation oscillations are studied and explicit results derived for two modes. The condition for internal resonance and universal power spectral relations are derived in this section. In Section 3.3, the damping rates of the relaxation oscillations are studied and explicit expressions are derived for two modes. Finally, the results of this paper are summarized in the conclusion.

2 Steady state solutions

The steady state solutions of the global rate equations (1, 2) are given by the implicit equations

$$1 = \frac{\gamma_p}{L}\int_0^L f_p(z)\overline{\mathcal{J}}(z)\mathrm{d}z, \tag{3}$$

$$w(z) = \overline{\mathcal{J}}(z)\left\{1 + \sum_p f_p(z)\overline{\mathcal{I}}_p\right\}. \tag{4}$$

An overbar indicates that the function is evaluated in steady state. Little can be said without any additional information. Therefore, we particularize the analysis of this section to a linear Fabry-Perot for which $f_p(z) = 2\sin^2(k_p z)$. We adopt the notation convention $f_{n,p} = 2\sin^2(nk_p z)$.

2.1 Single mode

The implicit single mode intensity equation is

$$1 = \frac{1}{L}\int_0^L \frac{f_p(z)w(z)}{1+\overline{\mathcal{I}}_p f_p(z)}\mathrm{d}z. \tag{5}$$

If the pump profile is constant, the exact solution of equation (5) is

$$\overline{\mathcal{I}}_p = \frac{1}{4}\left(4w - 1 - \sqrt{8w+1}\right). \tag{6}$$

To solve equation (5) if w is not constant, a first approach is to expand the denominator in powers of $\overline{\mathcal{I}}_p f_p$ and to use the recurrence relation

$$f_p^n = \frac{1}{2^{n-1}}\sum_{k=0}^{n-1}(-1)^{n+1-k}\binom{2n}{k} f_{n-k,p} \tag{7}$$

to generate the series expansion

$$1 = \sum_{n=0}^{\infty}(\overline{\mathcal{I}}_p/2)^n \sum_{k=0}^{n}(-1)^k\binom{2n+2}{k} w_{n-k+1,p} \tag{8}$$

$$= w_p + \frac{1}{2}\overline{\mathcal{I}}_p\left(w_{2,p} - 4w_p\right) + \frac{1}{4}\overline{\mathcal{I}}_p^2\left(w_{3,p} - 6w_{2,p} + 15w_p\right) + \cdots, \tag{9}$$

with the definition of the modal pump averages

$$w_{n,p} \equiv \frac{1}{L}\int_0^L w f_{n,p}\mathrm{d}z = \frac{\int_0^L \phi^*(nk_p z)w(z)\phi(nk_p z)\mathrm{d}z}{\int_0^L |\phi(nk_p z)|^2\,\mathrm{d}z}. \tag{10}$$

Since the index 1 is physically irrelevant, we write w_p instead of $w_{1,p}$. It is clear that $w_{n,p}$ is the average of the pump profile $w(z)$ over the nth harmonic of the single lasing cavity mode. A series reversion yields the solution

$$\overline{\mathcal{I}}_p = 2\frac{w_p - 1}{4w_p - w_{2,p}}\mathcal{K}_p, \tag{11}$$

$$\mathcal{K}_p = 1 + \left(15w_p - 6w_{2,p} + w_{3,p}\right) y_p + O(y_p^2),$$

where $\mathcal{K}_p$ is a series in powers of

$$y_p = (w_p - 1)\,/\,(w_{2,p} - 4w_p)^2.$$

It is a simple matter to keep as many terms as needed in equation (9) and to reverse the series to obtain the intensity to any desired accuracy.

Another way to analyze the implicit equation (5) for the steady state intensity is to sum in closed form the coefficients of the modal pump coefficients in the series (8)

$$1 = 4\sum_{k=0}^{\infty}\frac{\left(2\overline{\mathcal{I}}_p\right)^k w_{k+1,p}}{\sqrt{1+2\overline{\mathcal{I}}_p}\left(1+\sqrt{1+2\overline{\mathcal{I}}_p}\right)^{2k+2}}. \tag{12}$$

We use this result to study in more details some aspects of the nonlinear resonator dynamics.

One way to use equation (12) is to evaluate the contribution of w_p alone to the intensity by setting $w_{k+1,p} = w_p\delta_{k,0}$ in equation (12). Let $\mathcal{D}_p$ be that contribution. It is given by

$$4w_p = \sqrt{1+2\mathcal{D}_p}\left(1+\sqrt{1+2\mathcal{D}_p}\right)^2.$$

Close to threshold, $w_p \to 1$ and $\mathcal{D}_p \to (w_p - 1)/2$. From equation (11) we find that in the same limit $\overline{\mathcal{I}}_p \to 2\,(w_p-1)\,/\,(4w_p - w_{2,p}) = 2\,(w_p - 1)\,/\,(4 - w_{2,p})$. Combining these results leads to

$$\lim_{w_p\to 1}\mathcal{D}_p/\overline{\mathcal{I}}_p = 1 - w_{2,p}/4.$$

For a constant pump, $w_{n,p} = w$ and $\lim_{w_p\to 1}\mathcal{D}_p/\overline{\mathcal{I}}_p = 3/4$. In the large intensity limit, $2\mathcal{D}_p \to (4w_p)^{2/3}$ and $\overline{\mathcal{I}}_p \to w_p$ leading to

$$\lim_{w_p\to\infty}\mathcal{D}_p/\overline{\mathcal{I}}_p = (2/w_p)^{1/3}.$$

This shows that the contribution of the cavity mode $\phi(k_p z)$ to the steady state intensity decreases from $1 - w_{2,p}/4$ to zero as the intensity varies from zero to infinity.

It is therefore a poor approximation to neglect in a Fabry-Perot the harmonics of the single lasing mode generated by the nonlinear light-matter coupling, and the approximation gets worse as the intensity increases.

Another way to analyze the expansion (12) is to define the pump gratings

$$W_{n,p} = \frac{1}{L}\int_0^L w(z)\cos(2nk_p z)\,\mathrm{d}z.$$

Note that $W_{0,p} \equiv W_0$. Equation (12) becomes

$$1 = A_p W_0 - 4\sum_{k=0}^{\infty} B_{k,p} W_{k+1,p} \tag{13}$$

$$A_p = \frac{2}{\sqrt{1+2\overline{\mathcal{I}}_p}\left(1+\sqrt{1+2\overline{\mathcal{I}}_p}\right)},$$

$$B_{k,p} = \frac{\left(2\overline{\mathcal{I}}_p\right)^k}{\sqrt{1+2\overline{\mathcal{I}}_p}\left(1+\sqrt{1+2\overline{\mathcal{I}}_p}\right)^{2k+2}}.$$

In this expansion, the fundamental mode and all the harmonics contribute to A_p while only departures from the constant pump contribute to the infinite sum *via* the pump grating coefficients $W_{k,p}$. In the low intensity limit $\overline{\mathcal{I}}_p \to 0$, the coefficients A_p and $B_{k,p}$ are

$$A_p \to 1, \qquad B_{k,p} \to \frac{1}{4}\left(\overline{\mathcal{I}}_p/2\right)^k.$$

Thus at threshold the only nonvanishing pump grating weights are $A_p = 1$ and $B_{0,p} = 1/4$, and equation (13) reduces to the condition $w_p = 1$. The weight of all other pump gratings vanishes in this limit. This result is a justification, in steady state and close to the laser first threshold, of the TSD approximation scheme [2], though $W_{1,p}$ is neglected in the standard formulation of the TSD rate equations. The high intensity limit of the pump grating weights is

$$A_p \to 1/\overline{\mathcal{I}}_p, \quad B_{k,p} \to 1/\left(2\overline{\mathcal{I}}_p\right)^{3/2}.$$

Therefore, all coefficients $B_{k,p}$ vanish in that limit, being significantly nonzero only in the intermediate domain, in particular in the low but finite intensity domain. Note that in the large intensity limit, all pump gratings vanish with the same k-independent law. For $k > 0$, the functions $B_{k,p}$ have a maximum at

$$\overline{\mathcal{I}}_{k,p} = \frac{1}{9}\left[k^2 + 2k - 2 + (k+1)\left(k^2+2k+4\right)^{1/2}\right].$$

The first maxima are at

$$\overline{\mathcal{I}}_{1,p} = \frac{1}{9}\left(1+2\sqrt{7}\right) \simeq 0.699,$$

$$\overline{\mathcal{I}}_{2,p} = \frac{2}{3}\left(1+\sqrt{3}\right) \simeq 1.821,$$

$$\text{and} \quad \overline{\mathcal{I}}_{3,p} = \frac{1}{9}\left(13+4\sqrt{19}\right) \simeq 3.382.$$

Figure 1 displays the weights A_p and $B_{k,p}$ *versus* the intensity.

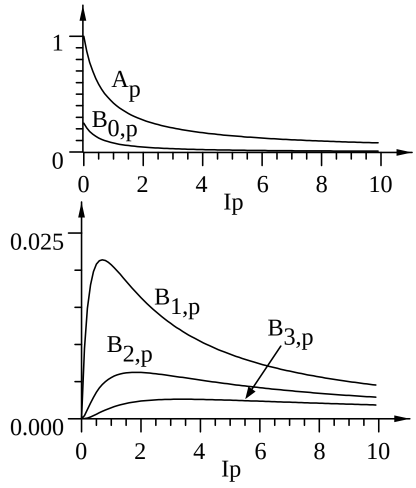

Fig. 1. Dependence of five lowest pump grating weights on intensity.

2.2 Two modes

The two-mode steady state modal intensities are determined by the two coupled implicit equations

$$1 = \frac{\gamma_p}{L}\int_0^L \frac{w(z) f(k_p z)}{1 + f(k_1 z)\overline{\mathcal{I}}_1 + f(k_2 z)\overline{\mathcal{I}}_2}\mathrm{d}z, \tag{14}$$

with $p = 1, 2$. For a linear Fabry-Perot, we have the recurrence relation (7) and

$$f(k_p z) f(k_q z) = f(k_p z) + f(k_q z) - \frac{1}{2} f(k_p z + k_q z) - \frac{1}{2} f(k_p z - k_q z). \tag{15}$$

Using these relations, we obtain from (14) the expansions

$$w_p - \gamma_p^{-1} = \overline{\mathcal{I}}_p\left(2w_p - \frac{1}{2}w_{2,p}\right) + \overline{\mathcal{I}}_q\left(w_p + w_q - \frac{1}{2}w_{p,+,q} - \frac{1}{2}w_{p,-,q}\right) + \ldots$$

with the notation $w_{np,\pm,mq} \equiv w\left(nk_p \pm mk_q\right)$. The solution of these two coupled equations is of the form $\overline{\mathcal{I}}_p = \mathcal{F}(p,q)$, with $p, q = 1, 2$ and $p \neq q$. The function $\mathcal{F}(p,q)$ is too complicated to be useful for analytic work. However, if $w_p = w$, it reduces to

$$\mathcal{F}(p,q) = \frac{2}{5}(3y_p - 2y_q) + \frac{4}{5^3}\left(41y_p^2 - 18y_p y_q - 9y_q^2\right) + \frac{2}{5^5}\left(881y_p^3 + 3318y_p^2 y_q - 4582 y_p y_q^2 + 1106 y_q^3\right) + O(y^4), \tag{16}$$

with $y_n = 1 - 1/(w\gamma_n)$ and using the fairly obvious notation $O(y^4) \equiv O(y_p^a y_q^b \delta_{a+b,4})$. In the double limit $\gamma_p = \gamma_q = \gamma$ and $w_p = w$, the modal intensities become

$$\overline{\mathcal{I}}_1 = \overline{\mathcal{I}}_2 = \frac{2}{5}y + \frac{56}{5^3}y^2 + \frac{1446}{5^5}y^3 + O(y^4). \tag{17}$$

The corresponding expressions for the TSD solutions and a discussion of the convergence of their series expansions are given in Appendix B.

2.3 The two-mode threshold

The threshold for two-mode oscillation derives from the simultaneous solution of the pair of equations (14) with $\overline{\mathcal{I}}_2 = 0$. A subscript "th" should be added to all functions and parameters in this section to clearly indicate that all the calculations are performed at the threshold. This will not be done for the sake of clarity. It is not difficult to prove with the help of the recurrence relations (7, 15) that the two threshold intensity equations are

$$\begin{aligned} 1 &= \frac{\gamma_2}{L}\int_0^L \frac{f_2(z)w(z)}{1+\overline{\mathcal{I}}_1 f_1(z)}\mathrm{d}z \\ &= \gamma_2 w_2 - 2\sum_{n=1}^{\infty}\frac{\left(2\overline{\mathcal{I}}_1\right)^n(\gamma_2/L)\int_0^L w(z)f(nk_1z)f(k_2z)\mathrm{d}z}{\sqrt{1+2\overline{\mathcal{I}}_1}\left(1+\sqrt{1+2\overline{\mathcal{I}}_1}\right)^{2n+2}}, \end{aligned} \tag{18}$$

$$\begin{aligned} 1 &= \frac{\gamma_1}{L}\int_0^L \frac{f_1(z)w(z)}{1+\overline{\mathcal{I}}_1 f_1(z)}\mathrm{d}z \\ &= 4\sum_{n=0}^{\infty}\frac{\left(2\overline{\mathcal{I}}_1\right)^n(\gamma_1/L)\int_0^L w(z)f\left(nk_1z+k_1z\right)\mathrm{d}z}{\sqrt{1+2\overline{\mathcal{I}}_1}\left(1+\sqrt{1+2\overline{\mathcal{I}}_1}\right)^{2n+2}}. \end{aligned} \tag{19}$$

In the simple case of constant pump $w_1 = w_2 = w$ and with the definitions $\gamma_1 = 1$, $\gamma_2 \equiv \gamma$ introduced without loss of generality, equation (18) becomes

$$\sqrt{1+2\overline{\mathcal{I}}_1}\left(1+\sqrt{1+2\overline{\mathcal{I}}_1}\right)^3 = 2\overline{\mathcal{I}}_1\gamma w/\left(\gamma w-1\right), \tag{20}$$

while equation (19) leads to the solution (6). The simultaneous solution of these equations is

$$\overline{\mathcal{I}}_1 = 4\gamma(\gamma w-1)\left(\gamma+2-2\gamma w\right)^{-2}, \tag{21}$$

together with the consistency condition

$$\left(4w-1-\sqrt{8w+1}\right)\left(\gamma+2-2\gamma w\right)^2 = 16\gamma(\gamma w-1). \tag{22}$$

Equations (22, 21) are the required results: they give the upper boundary in the (w,γ) plane for the stable single mode operation and the intensity of the first mode on that boundary. In Figure 2, this stability boundary is plotted in the (w,γ) plane as the curve labelled GLB. If γ is close to unity, the solution of equation (22) can be written as

$$\begin{aligned} w = 1 + \frac{6}{5}(1-\gamma) + \frac{26}{25}(1-\gamma)^2 \\ + \frac{654}{125}(1-\gamma)^3 + O\left[(1-\gamma)^4\right]. \end{aligned}$$

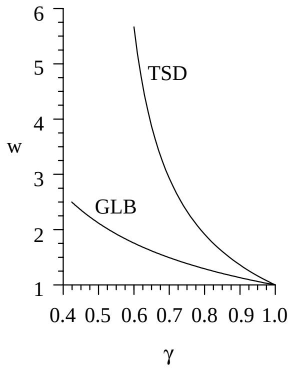

Fig. 2. Threshold for two-mode operation (*i.e.*, upper bound for stable single mode oscillation) in the (w,γ) plane. At the threshold, the intensity of the first mode is finite and the intensity of the second mode vanishes.

The corresponding upper boundary in the TSD approximation, derived in Appendix B, is

$$\begin{aligned} w_{\mathrm{TSD}} &= \left(-1+4\gamma-2\gamma^2\right)\left[\gamma(2\gamma-1)\right]^{-1} \\ &= 1-3\,(1-\gamma)+5\,(1-\gamma)^2 \\ &\quad -9\,(1-\gamma)^3+O\left[(1-\gamma)^4\right]. \end{aligned}$$

It is also plotted in Figure 2. There is a big difference between the two results: the TSD expression for the critical pump diverges at $\gamma = 1/2$ below which the critical pump becomes negative and the intensity is no longer real and positive. This singularity is an artefact of the truncated equations which is not found in the global equations for which a threshold exists for any positive γ. In the domain $0.5 < \gamma \le 1$, the TSD approximation of the threshold is systematically larger than the GLB result which diverges only at the origin: w increases monotonically from 1 to ∞ as γ decreases from 1 to 0.

A characterization of the two-mode threshold is important since it determines the domain in which a laser operates stably on a single mode. The problem of determining the limits imposed by the population inversion grating (*i.e.*, spatial hole burning) on the single-mode operation was analyzed in [20]. However, the result obtained in that paper relies on an inconsistent use of the assumption of slow spatial variation of the pump compared with cavity mode variations. This leads to equation (6) for the single mode intensity equation with w replaced by the space average of $w(z)$, but fails to reproduce equation (20) for the modal intensity at the lower boundary of the two-mode regime. The case of end-pumping was considered in [21] though with the same shortcoming as in [20].

3 The linearized dynamical equations

Linearizing equations (1, 2) with respect to deviations $\{x_p\}$ from the steady state leads to a set of N linear homogeneous equations

$$\begin{aligned} \lambda^2 x_p = &-\frac{\kappa_p\gamma_p\overline{\mathcal{I}}_p}{L}\int_0^L \frac{wf_p\sum_q f_q x_q}{1+\sum_q f_q\overline{\mathcal{I}}_q}\mathrm{d}z \\ &+\frac{\kappa_p\gamma_p\overline{\mathcal{I}}_p}{L}\int_0^L \frac{wf_p\sum_q f_q x_q}{\lambda+1+\sum_q f_q\overline{\mathcal{I}}_q}\mathrm{d}z \end{aligned} \tag{23}$$

where λ is the characteristic root.

3.1 Single mode regime

In the single mode regime, $\gamma_p = 1$ and the characteristic equation is

$$\lambda^2 = -\kappa_p (w_p - 1) + \frac{\kappa_p \overline{\mathcal{I}}_p}{L} \int_0^L \frac{w f_p^2}{\lambda + 1 + f_p \overline{\mathcal{I}}_p} \mathrm{d}z. \quad (24)$$

To solve this integral equation, we use the fact that κ_p is a large number, being typically in the range 10^4 to 10^6. In the range of pump parameters such that $w_p - 1 = \mathcal{O}(1)$, the characteristic equation has solutions in the form of a series in powers of $\kappa_p^{1/2}$. It is easy to solve equation (24) by iteration to obtain

$$\lambda_\pm = \pm \mathrm{i} \sqrt{\kappa_p (w_p - 1)} - \frac{\overline{\mathcal{I}}_p}{2} \frac{w_{pp}}{w_p - 1} + O(\kappa_p^{-1/2}),$$

where the symbol $w_{p \ldots p}$ with n indices is defined as $w_{p \ldots p} = (1/L) \int_0^L w(z) f_p^n \mathrm{d}z$. Let us emphasize that the expression for $\lambda_\pm$ is independent of both longitudinal pump profile and explicit cavity mode properties.

The characteristic roots $\lambda_\pm$ have been obtained with the assumption that $w_p - 1$ is finite. Close to the lasing first threshold, a different analysis is required. We define the vicinity of the threshold by $w_p - 1 = \alpha/\kappa_p$ where $\alpha = O(\kappa_p^0)$. In that range, the characteristic equation has two real solutions

$$\lambda_1 = -\kappa_p (w_p - 1) + O(\kappa_p^{-1}),$$
$$\lambda_2 = -\overline{\mathcal{I}}_p w_{ppp} / w_{pp} + O(\kappa_p^{-2}). \quad (25)$$

Here again, the result holds for any pump profile $w(z)$ and any cavity modes f_p. Thus, close to threshold the steady state does not display relaxation oscillations: perturbations from the steady state are simply damped. Though the two real roots (25) vanish at threshold, they have different orders of magnitude: $\lambda_1 = O(\kappa_p^0)$ while $\lambda_2 = O(\kappa_p^{-1})$.

Using the property $w_{pp} = 2w_p - (1/2) w_{2,p}$ for a linear Fabry-Perot, we get for the complex characteristic roots $\mathrm{Re}(\lambda_\pm) = (1/2)\mathcal{K}_p$ where $\mathcal{K}_p$ is defined by equation (11). Hence $\mathcal{K}_p$ is the width of the peak at the frequency $|\mathrm{Im}(\lambda_\pm)|$ in the power spectrum of the laser output. This power spectrum is generated, for instance, by the noise in the laser. For a linear Fabry-Perot, the root λ_2 can be expressed in terms of the modal pump averages as

$$\lambda_2 = -(w_p - 1) \left(\frac{15}{4} w_p - \frac{3}{2} w_{2,p} + \frac{1}{4} w_{3,p} \right) \times \left(2w_p - \frac{1}{2} w_{2,p} \right)^{-2} + O(\kappa_p^{-2}).$$

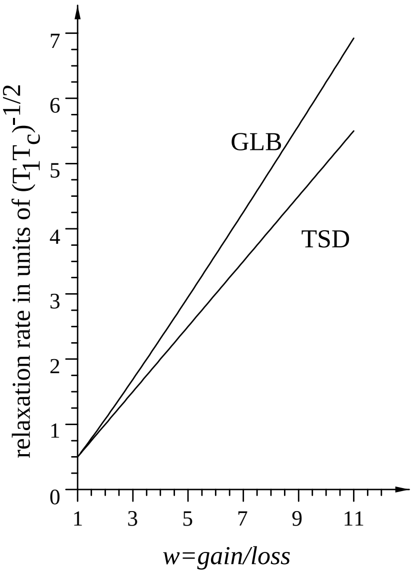

Fig. 3. Decay rate of the single mode relaxation oscillations.

If the pump is constant, the decay rates and the relaxation oscillation frequency are

$$\lambda_\pm = \pm \mathrm{i} \sqrt{\kappa_p (w - 1)} - \frac{3w\overline{\mathcal{I}}_p}{4(w-1)} + O(\kappa_p^{-1/2}), \quad (26)$$
$$\lambda_1 = -\kappa_p (w - 1) + O(\kappa_p^{-1}),$$
$$\lambda_2 = -10(w-1)/9 + O(\kappa_p^{-2}),$$

with the steady state intensity given by (6). The TSD analog of the characteristic roots (26) is

$$\lambda_{\pm \mathrm{TSD}} = \pm \mathrm{i} \sqrt{\kappa_p (w - 1)} - w/2 + O(\kappa_p^{-1/2}).$$

The relaxation oscillation frequencies coincide if the pump profile is a constant, which requires of course that the cavity be filled by the amplifying medium. However, the decay rate of these relaxation oscillation frequencies are quite different. They coincide at threshold, but the power broadening is always underestimated by the TSD theory as clearly shown in Figure 3 where the result $3w\overline{\mathcal{I}}_p/4(w - 1)$ and the TSD approximation are displayed.

3.2 Relaxation oscillation frequencies

If more than one mode is excited, we study separately the relaxation oscillations and their damping. This is suggested by the result obtained in the previous section, where it is clear that oscillations and damping operate on different time scales.

$$\Theta_1^2 = \frac{1}{10}\left(3y_p + 3y_q + R\right) + \frac{2621(y_p^3 + y_q^3) - 2437 y_p y_q \left(y_q + y_p\right) + (127y_p^2 + 127y_q^2 - 96y_p y_q)R}{2(5^3)R} + O(y^3),$$

3.2.1 Arbitrary mode number

In the large κ limit, equation (23) is approximated by

$$\lambda^2 x_p \simeq -\frac{\kappa_p\gamma_p\overline{\mathcal{I}}_p}{L}\sum_q x_q \int_0^L \frac{w f_p f_q}{1+\sum_q f_q \overline{\mathcal{I}}_q}\mathrm{d}z = -\sum_q a_{pq}x_q, \quad (27)$$

$$a_{pq} = \frac{\kappa_p\gamma_p\overline{\mathcal{I}}_p}{L}\int_0^L \overline{\mathcal{J}} f_p f_q \mathrm{d}z, \quad (28)$$

provided that all eigenvectors x_q have the same scaling in terms of κ. However, from equation (4), we have

$$\sum_{p\neq q}\overline{\mathcal{I}}_p \frac{1}{L}\int_0^L \overline{\mathcal{J}} f_p f_q \mathrm{d}z = w_q - (1+\Omega_q^2)/\gamma_q,$$

where the auxiliary frequency Ω_p is defined through

$$\Omega_p^2 = \frac{\gamma_p\overline{\mathcal{I}}_p}{L}\int_0^L \overline{\mathcal{J}} f_p^2 \mathrm{d}z = a_{pp}/\kappa_p.$$

Finally, we get a set of linear inhomogeneous equations for the coefficients a_{pq}

$$\sum_{p\neq q}\frac{\gamma_q}{\gamma_p\kappa_q}a_{pq} = \Delta_q \equiv \gamma_q w_q - 1 - \Omega_q^2. \quad (29)$$

These relations decrease the number of parameters since they allow to express the off-diagonal a_{pq} in terms of the diagonal $\{a_{pp}\}$ and the $\{w_p\}$. By definition $a_{pq} > 0$ and therefore $\Delta_q > 0$. This gives an upper bound to the auxiliary frequencies since $\Delta_q > 0$ implies $\Omega_q^2 < \gamma_q w_q - 1$.

3.2.2 Two modes

The linear relations equation (29) yield $a_{pq} = \gamma_p\kappa_p\Delta_q/\gamma_q$ and the characteristic equation derived from equation (27) becomes $\lambda^4 + b\lambda^2 + c = 0$ with

$$b = \kappa_1\Omega_1^2 + \kappa_2\Omega_2^2, \qquad c = \kappa_1\kappa_2\left(\Omega_1^2\Omega_2^2 - \Delta_1\Delta_2\right).$$

The auxiliary functions Ω_p are explicitly evaluated in Appendix C. The roots of the characteristic equation are

$$\lambda_1^2 = -\frac{1}{2}\left\{\kappa_1\Omega_1^2 + \kappa_2\Omega_2^2 + \sqrt{\left(\kappa_1\Omega_1^2 - \kappa_2\Omega_2^2\right)^2 + 4\kappa_1\kappa_2\Delta_1\Delta_2}\right\} \equiv -\kappa_1\Theta_1^2, \quad (30)$$

$$\lambda_2^2 = -\frac{1}{2}\left\{\kappa_1\Omega_1^2 + \kappa_2\Omega_2^2 - \sqrt{\left(\kappa_1\Omega_1^2 - \kappa_2\Omega_2^2\right)^2 + 4\kappa_1\kappa_2\Delta_1\Delta_2}\right\} \equiv -\kappa_2\Theta_2^2. \quad (31)$$

The Θ_p are the relaxation oscillation frequencies in the two mode regime. The corresponding eigenvectors have components

$$\frac{x_2}{x_1} = -\frac{\lambda^2 + a_{11}}{a_{12}} = -\gamma_2\left(\lambda^2 + \kappa_1\Omega_1^2\right)/\left(\kappa_1\Delta_1\gamma_1\right).$$

Let $\delta = \kappa_1\Omega_1^2 - \kappa_2\Omega_2^2$. The eigenvectors can be written as

$$\begin{pmatrix} x_{11} \\ x_{21}\end{pmatrix} = \frac{1}{N_1}\begin{pmatrix} 2\kappa_1\Delta_1\gamma_1/\gamma_2 \\ -\delta + \sqrt{\delta^2 + 4\kappa_1\kappa_2\Delta_1\Delta_2}\end{pmatrix} \quad \text{for} \quad \lambda = \lambda_1, \quad (32)$$

$$\begin{pmatrix} x_{12} \\ x_{22}\end{pmatrix} = \frac{1}{N_2}\begin{pmatrix} 2\kappa_1\Delta_1\gamma_1/\gamma_2 \\ -\delta - \sqrt{\delta^2 + 4\kappa_1\kappa_2\Delta_1\Delta_2}\end{pmatrix} \quad \text{for} \quad \lambda = \lambda_2, \quad (33)$$

where N_1 and N_2 are normalization constants.

In the limit $\kappa_p = \kappa_q$ and for a Fabry-Perot with $w_p = w$, the power expansion of the two frequencies Θ_p^2 are

see equation above

and $\Theta_2^2(R) = \Theta_1^2(-R)$ where

$$R = \sqrt{129y_p^2 - 242y_p y_q + 129y_q^2}$$

and $y_n = 1 - 1/\left(w\gamma_n\right)$. If, in addition, the linear gains are equals $\gamma_p = \gamma_q = \gamma$, the Θ_p^2 become

$$\Theta_1^2 = y + y^2 + y^3 + O(y^4),$$
$$\Theta_2^2 = \frac{1}{5}y + \frac{33}{5^3}y^2 + \frac{853}{5^5}y^3 + O(y^4).$$

In the same flat gain/loss limit ($\kappa_p = \kappa_q = \kappa$, $\gamma_p = \gamma_q = \gamma$) and with constant pump profile ($w_p = w$), we obtain the following expansions in terms of γw

$$\Theta_1^2 = \gamma w - 1, \quad (34)$$

$$\Theta_2^2 = \frac{1}{5}\left(\gamma w - 1\right) + \frac{2^3}{5^3}\left(\gamma w - 1\right)^2 - \frac{172}{5^5}\left(\gamma w - 1\right)^3 + ...$$

and the eigenvectors are

$$\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which is an alternative way to express the antiphase theorem [19]. In the same limit, the TSD model yields the frequencies

$$\Theta^2_{1,\mathrm{TSD}} = \gamma w - 1,$$
$$\Theta^2_{2,\mathrm{TSD}} = \frac{1}{5}(\gamma w - 1) + \frac{2^3}{5^3}(\gamma w - 1)^2 - \frac{72}{5^5}(\gamma w - 1)^3 + ...$$

Again, this expansion is close to the exact result (34) if γw is close to unity, but it suffers from the defect that the TSD expansion is the same whatever the longitudinal profile is.

Note that in the flat gain/loss limit and with constant pump profile (which implies a completely filled Fabry-Perot cavity), the parameters γ_p, κ_p, and w_p are independent of their index. Therefore Ω_p is also index-independent, which implies

$$\delta = 0, \qquad \Theta^2_1 = \gamma w - 1, \qquad \Theta^2_2 = 2\Omega^2 - \Theta^2_1. \quad (35)$$

3.2.3 Internal resonance

We know from studies of the TSD equations in the flat gain/loss limit that in the two-mode regime there is a special value of the pumping rate, $w_r = 15/7 \simeq 2.143$, for which there is an internal resonance: $\Theta^2_{1,\mathrm{TSD}} = 4\Theta^2_{2,\mathrm{TSD}}$ [22]. The same phenomenon occurs with the global rate equations. Using the expansion of Θ_j in powers of y with flat gain/loss distribution and $w_p = w$, it can be shown that the internal resonance $\Theta^2_1 = 4\Theta^2_2$ occurs for $y_r \simeq 0.712$ or $w_r \simeq 3.472$ which deviates from the TSD result by more than 50%.

3.2.4 Universal power spectral identities

Using the eigenvectors (32, 33), the modal intensities are given by

$$\mathcal{I}_p(\tau) = \overline{\mathcal{I}}_p + \varepsilon \sum_q \left(\mathcal{C}_q x_{pq} \mathrm{e}^{-\mathrm{i}\Theta_q \tau} + \mathrm{c.c.}\right) + O(\varepsilon^2),$$

and the coefficients $\{\mathcal{C}_p\}$ are determined by the initial condition $\mathcal{I}_p(0)$. The total intensity is

$$\mathcal{I}_{\mathrm{TOT}} = \sum_p \mathcal{I}_p$$
$$= \sum_p \overline{\mathcal{I}}_p + \varepsilon \sum_p \left(\mathcal{C}_p \mathrm{e}^{-\mathrm{i}\Theta_p \tau} \sum_q x_{qp} + \mathrm{c.c.}\right) + O(\varepsilon^2).$$

For two modes, this expression reduces to

$$\mathcal{I}_{\mathrm{TOT}} = \overline{\mathcal{I}}_1 + \overline{\mathcal{I}}_2 + \varepsilon\left[\mathcal{C}_1 (x_{11} + x_{21}) \mathrm{e}^{-\mathrm{i}\Theta_1 \tau} + \mathrm{c.c.}\right]$$
$$+ \varepsilon\left[\mathcal{C}_2 (x_{12} + x_{22}) \mathrm{e}^{-\mathrm{i}\Theta_2 \tau} + \mathrm{c.c.}\right] + O(\varepsilon^2).$$

Since $\lambda^2_p < 0$, the eigenvalues are purely imaginary at this order. The power spectral density of $\mathcal{I}_{\mathrm{TOT}}$ at frequency Θ_q is $|\mathcal{C}_q (x_{1q} + x_{2q})|^2$. The power spectral density of $\mathcal{I}_p$ at the frequency Θ_q is $|\mathcal{C}_q x_{pq}|^2$. Let $P(\mathcal{X}, p)$ be the power spectral density of the dynamical variable $\mathcal{X}$ at the frequency Θ_p. Using the expressions (32, 33) and the fact that $\Delta_p > 0$, we obtain

$$P(\mathcal{I}_{\mathrm{TOT}}, 1) = \left[\sqrt{P(\mathcal{I}_1, 1)} + \sqrt{P(\mathcal{I}_2, 1)}\right]^2, \quad (36)$$

$$P(\mathcal{I}_{\mathrm{TOT}}, 2) = \left[\sqrt{P(\mathcal{I}_1, 2)} - \sqrt{P(\mathcal{I}_2, 2)}\right]^2. \quad (37)$$

These relations are universal in the sense that they do not depend on the coefficients $\{\mathcal{C}_p\}$ which contain all the information on the initial condition. They are also independent of any of the laser operating parameters. The only limitation is the validity of the linearized dynamical equations to describe the laser. Let us stress that the relations (36, 37) have been derived for arbitrary $w(z)$ and f_p. Similar relations were derived for specific models, cavities and constant pumping rates, and confirmed experimentally in [23].

3.3 Damping rates

3.3.1 Arbitrary mode number

For an arbitrary mode number, a study of the damping rates has to start with equation (23):

$$\lambda^2 x_p + \sum_q \left[a_{pq} - \alpha_{pq}(\lambda)\right] x_q = 0, \quad (38)$$

with a_{pq} defined by equation (28) and

$$\alpha_{pq}(\lambda) = \frac{\kappa_p \gamma_p \overline{\mathcal{I}}_p}{L} \int_0^L \frac{w f_p f_q}{\lambda + w/\overline{\mathcal{J}}} \mathrm{d}z.$$

A property that will be used in the following is

$$\lim_{\lambda \to \infty} \alpha_{pq}(\lambda) = \frac{d_{pq}}{\lambda} + O(1/\lambda^2),$$
$$d_{pq} = \frac{\kappa_p \gamma_p \overline{\mathcal{I}}_p}{L} \int_0^L w f_p f_q \mathrm{d}z.$$

An approximation scheme to solve equation (38) can be set up as follows. The coefficients $a_{pq} - \alpha_{pq}$ have the property that a_{pq} is proportional to κ while α_{pq} can be expressed as a series in descending powers of $\kappa^{1/2}$ starting with a $\kappa^{1/2}$ term. It follows from this structure that the eigenvalues will be of the form $\lambda = \lambda_1 \kappa^{1/2} + \lambda_0 + \lambda_{-1} \kappa^{-1/2} + O(1/\kappa)$. From the structure of the characteristic equation (23), it also follows that λ_1 is imaginary and the next order in κ will give a real coefficient λ_0 which is the dominant order contribution to the damping of the relaxation oscillation frequency. This conclusion holds only if λ_1 is $O(\kappa^0)$. A necessary condition is therefore that the laser is not operating too close to a threshold where a new mode has emerged. For instance, for the simplified model with flat gain/loss distribution and with constant pump, there is a very thin boundary layer $\gamma w - 1 = O(1/\kappa)$ where a different asymptotic analysis is required, which yields two real roots instead of a pair of complex conjugate roots as was shown in Section 3.1. That boundary layer is excluded from the following analysis.

3.3.2 Two modes

In the two-mode regime, the linearized equations for the x_p, equation (38), lead to the biquadratic characteristic equation $\lambda^4 + \lambda^2(b_{11} + b_{22}) + b_{11}b_{22} - b_{12}b_{21} = 0$ where

$$b_{pp} = \kappa_p \Omega_p^2 - \alpha_{pp}, \qquad b_{pq} = \gamma_p \kappa_p \Delta_q/\gamma_q - \alpha_{pq}.$$

The formal solution of the biquadratic, which is still an integral equation for λ, is

$$2\lambda^2 = -\kappa_1 \Omega_1^2 - \kappa_2 \Omega_2^2 + \alpha_{11} + \alpha_{22} \pm \sqrt{\delta^2 + 4a_{12}a_{21}} \sqrt{1 + \frac{L_1(\alpha) + L_2(\alpha)}{\delta^2 + 4a_{12}a_{21}}}, \quad (39)$$

where $L_1(\alpha) = 2\delta(\alpha_{22} - \alpha_{11}) - 4(a_{12}\alpha_{21} + a_{12}\alpha_{21})$ and $L_2(\alpha) = 4\alpha_{12}\alpha_{21}$ are, respectively, linear and quadratic in α_{pq}. The dominant orders of magnitude of the various contributions to equation (39) are: $\lambda \sim \kappa^{1/2}, a \sim \kappa, \alpha \sim \kappa^{1/2}, \delta \sim \kappa, L_1 \sim \kappa^{3/2}$, and $L_2 \sim \kappa$. With this, we can derive the explicit result

$$\lambda_p = \pm \mathrm{i}\sqrt{\kappa_p}\Theta_p - D_p + O(\kappa_p^{-1/2}),$$

where the damping rates are

$$D_p = \frac{d_{11} + d_{22}}{4\kappa_p \Theta_p^2} + (-1)^p \frac{\delta(d_{22} - d_{11}) - 2(d_{12}a_{21} + d_{21}a_{12})}{4\kappa_p \Theta_p^2 \sqrt{\delta^2 + 4a_{12}a_{21}}} \cdot \quad (40)$$

It is difficult to obtain general properties of these damping rates. However, if $w(z) > 0$, there is an obvious conclusion for the two-mode regime since $\left(\delta^2 + 4a_{12}a_{21}\right)^{1/2} \pm \delta$ is always positive because $a_{12}a_{21} > 0$. From $a_{pq} > 0$ and $d_{pq} > 0$, it then follows that $D_1 > 0$ and the only possibility of an instability is $\mathrm{Re}(\lambda_2) = 0$.

In the limit of flat gain/loss distributions ($\gamma_p = \gamma_q$, $\kappa_p = \kappa_q$) and constant longitudinal pump profile ($w_p = w$), it is easy to verify that $a_{pq} = \kappa\Delta = \kappa\left(\gamma w - 1 - \Omega^2\right)$, $d_{pq} = \kappa\gamma w \overline{\mathcal{I}}$, and $d_{pp} = d_{qq} = (3/2)d_{pq}$. Using (35) this leads to

$$D_1 = \frac{5}{2}\frac{\gamma w}{\gamma w - 1}\overline{\mathcal{I}}, \qquad D_2 = \frac{1}{2}\frac{\gamma w}{2\Omega^2 - \gamma w - 1}$$

from which it follows that

$$D_1/D_2 = 1 + \frac{2^3}{5^2}(\gamma w - 1) - \frac{172}{5^4}(\gamma w - 1)^2 + \cdots = 1 + \frac{2^3}{5^2}y + \frac{28}{5^4}y^2 + \cdots$$

In this limit, no instability is possible in the rate equation limit.

4 Pump averages

Practically all results derived in this paper depend on the pump averages (10). In this section we evaluate these averages in three classic situations. For an end-pumped linear Fabry-Perot $f_p = 2\sin^2(k_p z)$ with an exponential decrease of the normalized pump $w(z) = w\alpha L \exp(-\alpha z)/\left[1 - \exp(-\alpha L)\right]$ where w is the space average of the pump profile across the cavity, w_p is

$$w_p = \frac{1}{L}\int_0^L w f_p \mathrm{d}z = \frac{w}{1 + (\alpha/2k_p)^2}\cdot$$

Thus, since $\alpha/2k_p \ll 1$ in normal operating conditions, $w_p \simeq w$ and a single mode laser with end-pumping will have the same properties as a laser with constant pumping w. The same conclusion was obtained by different means in [24]. Another contact with a classic problem is the role of the filling factor with constant pumping $w(z) = w$ in the region ℓ but $w = \mathcal{J} = 0$ in the remainder of the cavity, *i.e.*, ℓ_e on one side of the amplifying medium and $L - \ell - \ell_\mathrm{e}$ on the other side. Then

$$w_p = w_0\frac{\ell}{L} + \frac{w_0}{2Lk_p}\left[\sin(2k_p\ell_\mathrm{e}) - \sin(2k_p(\ell_\mathrm{e} + \ell))\right].$$

Let us consider a more general configuration: a linear Fabry-Perot of total length L filled with an amplifying medium of length $\ell < L$. The resonator is empty over a length ℓ_e on one side of the amplifying medium and a length $L - \ell - \ell_\mathrm{e} \geq 0$ on the other side. In the empty sections, the absence of material medium implies $w = \overline{\mathcal{J}} = 0$. In the amplifying medium, the pump decreases according to an exponential law $w(z) = w_0\exp(-\alpha z)$. Then

$$w_p = \frac{w_0}{\alpha L}\mathrm{e}^{-\alpha\ell_\mathrm{e}}\left(1 - \mathrm{e}^{-\alpha\ell}\right) + \frac{w_0}{\alpha L}\frac{\mathrm{e}^{-\alpha(\ell_\mathrm{e}+\ell)}}{1 + (2k_p/\alpha)^2} \times \left\{\cos\left[2k_p(\ell_\mathrm{e} + \ell)\right] - \frac{2k_p}{\alpha}\sin\left[2k_p(\ell_\mathrm{e} + \ell)\right]\right\} - \frac{w_0}{\alpha L}\frac{\mathrm{e}^{-\alpha\ell_\mathrm{e}}}{1 + (2k_p/\alpha)^2}\left[\cos(2k_p\ell_\mathrm{e}) - \frac{2k_p}{\alpha}\sin(2k_p\ell_\mathrm{e})\right].$$

It is easy to generalize this expression of w_p to a cavity including different amplifying elements with different lengths and different linear absorption coefficients separated by empty sections of arbitrary length and linear absorption.

5 Conclusion

One of the main interest of the results derived in this paper is the rare opportunity to test the implications of a modal truncation. It is surprising to note how few laser properties can be derived without specifying the cavity modes and/or the pump profile. Without any assumption on $w(z)$, f_p, and γ_p, an asymptotic expression for the single mode relaxation oscillation frequency and its damping has been obtained in Section 3.1. In the two-mode regime, we have derived expressions (30, 31) for the oscillation frequencies, expressions (40) for their damping rates, and the universal properties (36, 37) of the power spectra. These results rely only on the asymptotic property $\kappa_p \gg 1$. It also follows from this analysis that the number of characteristic roots for a laser oscillating on N modes is $2N$. This is at variance with the TSD result which predicts $2N+1$ roots in the same conditions. Thus the extra root is without physical meaning, being an artefact of the truncation procedure.

Assuming a linear Fabry-Perot, *i.e.*, $f_{n,p} = 2\sin^2(nk_p z)$, the single mode intensity in steady state can be expressed as an expansion in powers of $\left(w_p - \gamma_p^{-1}\right)/\left(w_{2,p} - 4w_p\right)^2$ or as the solution of an implicit equation containing a sum over the pump averages $w_{n,p}$ with coefficients that are simple irrational functions of the intensity. Nothing comparable has been achieved for the two-mode steady state modal intensities. However, assuming a linear Fabry-Perot and a constant longitudinal pump profile ($w_{n,p} = w$) drastically reduces the difficulty of the problem and leads to explicit answers for most problems in steady state. In particular, series have been generated for the steady state two-modes intensities in powers of either w or $y_p = 1 - 1/\gamma_p w$. Assuming further $\kappa_p = \kappa_q$ leads to explicit expressions for the relaxation oscillation frequencies and their damping rates in the two-mode regime.

In steady state, the global and modal rate equations yield very similar expressions for the modal intensities provided the pump profile is constant and the cavity is a Fabry-Perot completely filled by the amplifying medium. Any departure from these limitations does not affect the TSD result while it may deeply change the expressions derived from the global rate equations.

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Appendix A: Convergence of the single mode expansions

In general, it is not possible to derive exact explicit solutions of the laser equations and therefore series expansions are needed to proceed further in the analysis. Consider for instance the exact solution (6) obtained in the case of a linear Fabry-Perot with constant pump. It is natural to expand it in powers of w. This leads to a major difficulty since such an expansion has a finite and in fact rather small radius of convergence. To appreciate this difficulty, we consider the function $F(z) = 4z - 1 - (8z+1)^{1/2}$ in the complex z-plane. It has a branch point on the real axis at $z = -1/8$ and therefore the function is defined in the complex plane except for a cut on the real axis from $-1/8$ to $-\infty$. Hence, the expansion of $F(z)$ in powers of z around z_c on the real axis has a radius of convergence $R = z_c + 1/8$. In other terms, the expansion of $\overline{\mathcal{I}}_p$ in powers of w around the laser threshold $w_c = 1$ converges up to $w_{\max} = 17/8$. However, the more general analysis for an arbitrary pump profile suggests an alternative expansion. Indeed, the series expansion (11) leads in the limit of constant pump profile to the result

$$\overline{\mathcal{I}}_p = \frac{2}{3}y + \frac{20}{3^3}y^2 + \frac{190}{3^5}y^3 + O\left(y^4\right),$$

with $y = 1 - 1/w$. The transformation $w \to y$ is a conformal transformation which maps the real non negative axis $[0, +\infty]$ of the w-plane into the fundamental interval $[0, 1]$ on the real axis of the y-plane. Let us replace w by $1/(1-y)$ in the exact solution (6) and consider the function $G(z) = 4/(1-z) - 1 - [1 + 8/(1-z)]^{1/2}$ in the complex z-plane. This function has a pole and branching point at $z = 1$ and a branching point at $z = 9$. Hence the function $G(z)$ is defined in the complex plane cut along the real axis between $+1$ and $+9$ and an expansion of $G(z)$ around the origin has a radius of convergence $R = 1$. In physical terms, the expansion in powers of y of the steady state intensity converges for the whole domain of physical relevance.

It is instructive to compare these results with the power expansions of the TSD single mode intensity

$$\overline{\mathcal{I}}_{\mathrm{TSD}}(w) = -2 + w/2 + \sqrt{2 + (w/2)^2}. \tag{A.1}$$

In the complex w-plane, the function $\overline{\mathcal{I}}_{\mathrm{TSD}}(w)$ has a pair of branching points on the imaginary axis at $w = \pm 2\mathrm{i}\sqrt{2}$. Hence, the expansion of $\overline{\mathcal{I}}_{\mathrm{TSD}}(w)$ in powers of $w - w_c$ has a radius of convergence $R = \sqrt{8 + w_c^2}$, and the series converges up to $w_{\max} = w_c + R$ on the real axis. Hence, the expansion around $w_c = 1$ has a radius of convergence $R = 3$ and $w_{\max} = 4$. Applying the conformal mapping $y = 1 - 1/w$, the TSD single mode intensity is characterized by a pole and a branching point at $y = 1$ and two branching points at $y_\pm = 1 \pm \mathrm{i}/\sqrt{8}$. Therefore the series expansion around $y = 0$ has, again, a radius of convergence $R = 1$.

Appendix B: TSD solutions for two modes

In the TSD approximation, which implies $w_p = w$, and setting without loss of generality $\gamma_p = \gamma_1 = 1, \gamma_q = \gamma_2 \equiv \gamma$, the explicit solution of the TSD intensity equations in the two-mode regime is [1]

$$\overline{\mathcal{I}}_1 = \frac{2}{3}\frac{-4 + (2 - 3w)\gamma + \sqrt{(9w^2 - 8)\gamma^2 + 32\gamma - 8}}{3w\gamma - \sqrt{(9w^2 - 8)\gamma^2 + 32\gamma - 8}} \tag{B.1}$$

which is positive for

$$w \geq w_1 = \left(-2 + 4\gamma - \gamma^2\right)/\left[\gamma\left(2 - \gamma\right)\right],$$

and

$$\overline{\mathcal{I}}_2 = \frac{2}{3\gamma}\frac{2 - (4 + 3w)\gamma + \sqrt{(9w^2 - 8)\gamma^2 + 32\gamma - 8}}{3w\gamma - \sqrt{(9w^2 - 8)\gamma^2 + 32\gamma - 8}} \tag{B.2}$$

which is positive for

$$w \geq w_2 = \left(-1 + 4\gamma - 2\gamma^2\right) / \left[\gamma(-1 + 2\gamma)\right].$$

These expressions make sense only if the modal intensities are real and non negative. This constraints the gain ratio to the range $1/2 < \gamma \leq 1$. For $w < w_2$, $\overline{\mathcal{I}}_2$ is either negative or complex. Thus w_2 is identified as the TSD threshold of oscillation of the two-mode regime.

The convergence of series expansions of these modal intensities can be assessed easily. In the complex w-plane, both functions have a pair of purely imaginary branch points at $w_\pm = \pm \mathrm{i}\sqrt{8\left(4\gamma - \gamma^2 - 1\right)/9}$. Therefore, the radius of convergence for the expansions of $\overline{\mathcal{I}}_{1,\mathrm{TSD}}(w)$ and $\overline{\mathcal{I}}_{2,\mathrm{TSD}}(w)$ in powers of $w - w_\mathrm{c}$ is $R = \sqrt{w_\mathrm{c}^2 + 8\left(4\gamma - \gamma^2 - 1\right)/9}$ and the upper bound for convergence is $w_{\max} = w_\mathrm{c} + R$. For $\gamma = 1$, $w_\mathrm{c} = w_2 = 1$ and one finds $R = 5/3$ which yields $w_{\max} = 8/3$. This $w_{\max}$ is significantly smaller than the $w_{\max} = 4$ of the single mode TSD intensity.

Performing the conformal mapping $y = 1 - \alpha/w$ where α is real and positive, leads to an expression for $\overline{\mathcal{I}}_{1,\mathrm{TSD}}(y)$ and $\overline{\mathcal{I}}_{2,\mathrm{TSD}}(y)$ which has a pole and a branching point at $y = 1$ and two branching points at $y_\pm = 1 \pm 3\mathrm{i}\alpha\gamma/\sqrt{8\left(4\gamma - \gamma^2 - 1\right)}$. Thus, the series expansions of these two intensities in powers of y have a radius of convergence $R = 1$ which again covers the whole domain of physical relevance.

For the two-mode TSD modal intensities (B.1, B.2), an expansion in powers of $w - w_2$ yields

$$\overline{\mathcal{I}}_{1,\mathrm{TSD}} = 2\frac{1-\gamma}{2\gamma - 1} + \frac{2\gamma(2-\gamma)}{-1 + 4\gamma + 2\gamma^2}(w - w_2) + O\left[(w - w_2)^2\right],$$

$$\overline{\mathcal{I}}_{2,\mathrm{TSD}} = \frac{2(w - w_2)}{-1 + 4\gamma + 2\gamma^2} + O\left[(w - w_2)^2\right].$$

The function $\overline{\mathcal{I}}_{1,\mathrm{TSD}}(w = w_2) = 2(1-\gamma)/(2\gamma - 1)$ is of course the single mode intensity (A.1) for $\gamma_p w = w_2$. If $\gamma = 1$, then

$$\overline{\mathcal{I}}_{1,\mathrm{TSD}} = \overline{\mathcal{I}}_{2,\mathrm{TSD}} = \frac{2}{5}(w-1) + \frac{6}{5^3}(w-1)^2 - \frac{54}{5^5}(w-1)^3 + O\left[(w-1)^4\right].$$

Alternatively, we may expand the two modal intensities in powers of $y_2 = 1 - w_2/w$

$$\overline{\mathcal{I}}_{1,\mathrm{TSD}} = 2\frac{1-\gamma}{2\gamma-1} + 2y_2\frac{-2 + 9\gamma - 8\gamma^2 + 2\gamma^3}{(2\gamma - 1)\left(-1 + 4\gamma + 2\gamma^2\right)} + O(y_2^2),$$

$$\overline{\mathcal{I}}_{2,\mathrm{TSD}} = 2y_2\frac{-1 + 4\gamma - 2\gamma^2}{\gamma\left(-1 + 4\gamma + 2\gamma^2\right)} + O(y_2^2).$$

For $\gamma = 1$, this yields

$$\overline{\mathcal{I}}_{1,\mathrm{TSD}} = \overline{\mathcal{I}}_{2,\mathrm{TSD}} = \frac{2}{5}y + \frac{56}{5^3}y^2 + \frac{1496}{5^5}y^3 + O(y^4).$$

For the convergence of these series, it is relevant to notice that the w-expansions are alternate series since $0.5 \leq \gamma \leq 1$, while the y-expansions are positive series.

Appendix C: Evaluation of Ω_p^2

The evaluation of Ω_p^2 is central to the study of the linearized equations in the two-mode regime. By definition, we have for a Fabry-Perot resonator:

$$\begin{aligned}\Omega_p^2 &= \gamma_p \overline{\mathcal{I}}_p \frac{1}{L}\int_0^L \frac{w f_p^2}{1 + f_p\overline{\mathcal{I}}_p + f_q\overline{\mathcal{I}}_q}\mathrm{d}z \\ &= \gamma_p\overline{\mathcal{I}}_p \sum_{n=0}^{\infty}\frac{1}{L}\int_0^L w f_p^2\left(-f_p\overline{\mathcal{I}}_p - f_q\overline{\mathcal{I}}_q\right)^n \mathrm{d}z \\ &= \gamma_p\overline{\mathcal{I}}_p\sum_{n=0}^{\infty}(-)^n\sum_{k=0}^{n}\binom{n}{k}\overline{\mathcal{I}}_p^k\overline{\mathcal{I}}_q^{n-k}\frac{1}{L}\int_0^L w f_p^{k+2}f_q^{n-k}\mathrm{d}z.\end{aligned}$$

Using the relation (15), the first terms in the expansion of Ω_p^2 are

$$\begin{aligned}\Omega_p^2 &= \gamma_p\overline{\mathcal{I}}_p\left(w_{pp} - \overline{\mathcal{I}}_p w_{ppp} - \overline{\mathcal{I}}_q w_{ppq} + \overline{\mathcal{I}}_p^2 w_{pppp} + \overline{\mathcal{I}}_q^2 w_{ppqq} + 2\overline{\mathcal{I}}_p\overline{\mathcal{I}}_q w_{pppq} + ...\right) \\ &= \gamma_p\overline{\mathcal{I}}_p\left(2w_p - \frac{1}{2}w_{2,p}\right) - \gamma_p\overline{\mathcal{I}}_p^2\left(\frac{15}{4}w_p - \frac{3}{2}w_{2,p} + \frac{1}{4}w_{3,p}\right) \\ &\quad - \gamma_p\overline{\mathcal{I}}_p\overline{\mathcal{I}}_q\left(2w_p + \frac{3}{2}w_q - \frac{1}{2}w_{2,p} - w_{p,+,q} - w_{p,-,q} + \frac{1}{4}w_{2p,+,q} + \frac{1}{4}w_{2p,-,q}\right) + ...\end{aligned}$$

If $w_p = w$, we have

$$\frac{\Omega_p^2}{w\gamma_p} = \frac{3}{2}\overline{\mathcal{I}}_p - \frac{5}{2}\overline{\mathcal{I}}_p^2 - \frac{3}{2}\overline{\mathcal{I}}_p\overline{\mathcal{I}}_q + \frac{35}{8}\overline{\mathcal{I}}_p^3 + \frac{9}{4}\overline{\mathcal{I}}_p\overline{\mathcal{I}}_q^2 + 5\overline{\mathcal{I}}_p^2\overline{\mathcal{I}}_q + ...$$

Using the expression (16) for the modal intensities yields

$$\frac{\Omega_p^2}{w\gamma_p} = \frac{3}{5}\left(3y_p - 2y_q\right) - \frac{2}{5^3}\left(12y_p^2 - 51y_py_q + 37y_q^2\right) - \frac{2}{5^5}\left(2196y_p^3 - 6862y_p^2y_q + 6238y_py_q^2 - 1579y_q^3\right) + ...$$

In the further limit $\gamma_p = \gamma_q$, we obtain

$$\Omega_p^2 = w\gamma_p\overline{\mathcal{I}}_p\left(\frac{3}{2} - 4\overline{\mathcal{I}}_p + \frac{93}{8}\overline{\mathcal{I}}_p^2 + ...\right)$$

with $\overline{\mathcal{I}}_p$ given by equation (17) up to third order in y_p. Therefore a systematic expansion in powers of either y_p

or $w\gamma_p - 1$ is

$$\begin{aligned}\Omega_p^2 &= \frac{3}{5}y_p + \frac{79}{5^3}y_p^2 + \frac{1989}{5^5}y_p^3 + ...\\ &= \frac{3}{5}(\gamma_p w - 1) + \frac{4}{5^3}(\gamma_p w - 1)^2 - \frac{86}{5^5}(\gamma_p w - 1)^3 + ...\end{aligned}$$

References

1. P. Mandel, *Theoretical problems in cavity nonlinear optics* (Cambridge University Press, 1997).
2. C.L. Tang, H. Statz, G. deMars, J. Appl. Phys. **34**, 2289 (1963).
3. T. Baer, Opt. Lett. **12**, 392 (1987).
4. S. Ilchenko, P.S. Volikov, V.L. Velichansky, F. Treussart, V. Lefèvre-Seguin, J.-M. Raimond, S. Haroche, Opt. Commun. **145**, 86 (1998) and references quoted therein.
5. C. Gmachl, F. Capasso, E.E. Narimanov, J.U. Nöckel, A.D. Stone, J. Faist, D.L. Sivco, A.Y. Cho, Science **280**, 1556 (1998).
6. U. Vietze, O. Krauß, F. Laeri, G. Ihlein, F. Schüth, B. Limburg, M. Abraham, Phys. Rev. Lett. **81**, 4628 (1998).
7. V.V. Klimov, M. Ducloy, V.S. Letokhov, Phys. Rev. A **59**, 2996 (1999).
8. T. Harayama, P. Davies, K.S. Ikeda, Phys. Rev. Lett. **82**, 3803 (1999).
9. I. Protsenko, P. Domoko, V. Lefèvre-Seguin, J. Hare, J.M. Raimond, L. Davidovich, Phys. Rev. A **59**, 1667 (1999).
10. E.S. Kovalenko, Izvestiya VUZ. Radiofizika **10**, 1765 (1967) [Radiophys. Quant. Electron. **10**, 989 (1967)].
11. E.S. Kovalenko, A.V. Pugovkin, Izvestiya VUZ. Radiofizika **11**, 232 (1968) [Radiophys. Quant. Electron. **11**, 131 (1968)].
12. E.S. Kovalenko, L.I. Shangina, Izvestiya VUZ. Radiofizika **12**, 846 (1969) [Radiophys. Quant. Electron. **12**, 674 (1969)].
13. D. Pieroux, P. Mandel, Quant. Semiclass. Opt. **9**, L17 (1997).
14. L. Stamatescu, M.W. Hamilton (private communication).
15. K. Otsuka, R. Kawai, Y. Asakawa, P. Mandel, E.A. Viktorov, Opt. Lett. **23**, 201 (1998).
16. O.N. Evdokimova, L.N. Kaptsov, Kvant. Elektron. **16**, 1557 (1989) [Sov. J. Quant. Electron. **19**, 1001 (1989)].
17. N.B. Abraham, L. Sekaric, L.L. Carson, V. Seccareccia, P.A. Khandokhin, Ya.I. Khanin, I.V. Koryukin, V.G. Zhislina, Phys. Rev. A (in press).
18. G. Kozyreff, P. Mandel, Phys. Rev. A **58**, 4946 (1998).
19. P. Mandel, J. Opt. B: Quant. Semiclass. Opt. **1**, L1 (1999).
20. J.J. Zayhowski, Opt. Lett. **15**, 431 (1990).
21. S. Helmfrid, K. Tatsuno, J. Opt. Soc. Am. B **11**, 436 (1994).
22. D. Pieroux, T. Erneux, P. Mandel, Phys. Rev. A **54**, 3409 (1996).
23. P. Mandel, J.-Y. Wang, Phys. Rev. Lett. **75**, 1923 (1995) [correction: *ibid.* **76**, 1403 (1996)]; P. Mandel, K. Otsuka, J.-Y. Wang, D. Pieroux, Phys. Rev. Lett. **76**, 2694 (1996).
24. P.A. Khandokhin, E.A. Ovchinnikov, E.Yu. Shirokov (private communication).